

II. *An Account of a Book, intitl'd, P. D. Pauli Frisii Mediolanensis, &c. Disquisitio mathematica in causam physicam figuræ et magnitudinis Telluris nostræ ; printed at Milan in 1752. inscribed to the Count de Sylva, and consisting of Ten Sheets and a half in Quarto: By Mr. J. Short, F. R. S.*

Read Jan. 18, 1753. I. **I**T may be laid down as a rule in mix'd mathematics, " That the determination of no physical quantity be carried farther than the observations, or other mechanical measures, can bear ;" lest there follow this incongruity, of the conclusion being more extensive than the premises. It were absurd, for instance, in the resolution of a triangle, to compute an angle to the exactness of seconds, or a side to centesms of an inch, when, perhaps, the instruments used can measure no angle less than 10 minutes, or a side but to the exactness of a foot. The conclusions of arithmetic and geometry are indeed rigorously true, but they are only hypothetical ; and whenever the quantities, that enter any practical question, cannot be measured, but within certain limits, it were in vain to look for an answer perfectly accurate. The error of the instrument becomes itself one of the *data* ; and we must content ourselves to find the limits, which the quantity sought cannot well exceed, or fall short of, by such rules, as the great Mr. Cotes has left us in his excellent treatise on the subject.

2. In:

2. In like manner, when any physical theory is deduced from observations, its accuracy will still be in proportion to that of the observations, on which it is founded. Sir Isaac Newton, we find, in computing the ratio of the earth's axis to its equatorial diameter, confines himself to a reasonable approximation, and to three places of figures (229 to 230); because, whether that ratio is deduced from the different lengths of isochronous pendulums in different latitudes, from the measurement of distant degrees of a meridian, or from both; the elements of the calculus can scarcely furnish a greater degree of exactness. And of the same judicious caution we have many other examples in the works of that incomparable author.

3. On the other hand, when observations and theories are brought together, and compared, nothing can be justly inferred against a theory from its disagreement with the observations, unless that disagreement is greater, than can be fairly imputed to the imperfection of instruments, and to the unavoidable mistakes of an observer; especially, if the difference should be sometimes in excess, and at other times in defect; or, according to some of the observations, should intirely vanish.

4. Altho' these rules, manifestly well-founded, have been followed by all the best writers, our author observes, that several ingenious men, both in France, and in Italy, have deviated from them; particularly in treating of the famous question concerning the figure of the earth. Some, with Messieurs Clairaut and Bouguer, attributing too much to the observations, that have been made, and taking them
for

for absolutely exact, have concluded Sir Isaac Newton's reasonings on that subject to be faulty; while Father Boscowic, a Jesuit at Rome, making them quite loose and uncertain, thinks no argument at all can be drawn from them, concerning the earth's figure; far less in confirmation of the Newtonian theory.

5. In opposition to these two extremes, equally contrary to reason, as they are to each other, F. Frisi writes the treatise now before us; in the introduction to which he shews, 1. That, altho' the ratio of the axis of the earth to its equatorial diameter is, from M. de Maupertuis's operations in Lapland, and afterwards in France, that of 177 to 178; and by the theory only 229 to 230; yet the difference is no more, than what might arise from a mistake of about 60 toises in the measure of either of the two degrees, that are compared, or of 30 toises in each of them. Or, suppose the measure of the arcs to be exact, the same difference might be owing to an error of 4 or 5 seconds in the astronomical part. And such errors, or others equivalent to them, in a course of so many combined operations, our author looks upon as difficult to be avoided. But he adds, if the observations of M. de Maupertuis, and his fellow academicians, seem to differ from the theory, those of Messieurs Bouguer and de la Condamine exactly agree with it: According to whom, a degree at the equator, containing 56753 toises, and in latitude $49^{\circ} 22'$ 57183 toises, the difference of the axis and equatorial diameter comes out to be $\frac{1}{219}$.

6. In answer to F. Boscowic, and those who make no account of the observations, our author allows, that,

that, if they were such, as M. Caffini, and some other academicians, made in France, of the measure of a parallel of latitude, they could not be much depended on; that method being liable to several obvious inconveniencies. But he insists, that, with the excellent instruments, which were used, and, considering the distinguished skill of the observers, as well at the polar circle as in France, and at the equator, the error upon one degree of the meridian could not exceed 60 or 70 toises; which is a degree of exactness not only sufficient for the determination of the first question, *viz.* whether the spheroid of the earth is flat or long; but likewise to found an agreement between the observations and the theory, as near as can be expected or desired.

7. The work itself is divided into ten chapters :

- I. De observationibus circa telluris figuram hæcenus institutis.
- II. De principiis et hypothefibus quibusdam.
- III. De rotatione corporum, et vi centrifuga.
- IV. De mutationibus ex motu circulari ortis.
- V. De attractione corporum rotundorum.
- VI. De comparatione gravitatis in variis homogeneæ sphæroidis locis.
- VII. De figura terræ.
- VIII. De gradibus meridiani et parallelorum.
- IX. De loxodromiis nautarum, de parallaxi lunæ, et aliis ex eadem theoria pendentibus.
- X. De theoriæ et observationum consensu.

8. In chap. 1. we have a short history of the inquiries, that have been made into the magnitude and
figure

figure of the earth down to the present times ; and the preference is justly given to the measurements of Mr. Norwood in England, A. D. 1635, and of the members of the French Academy of Sciences since that time. From these he gathers, that, within less than 60 or 70 toises, the lengths of a degree of the meridian are as follows :

	°	'	Toises.
Lat.	0	0	56753
	45	0	57100
	49	22	57183
	53	0	57300
	66	20	57400

9. Chap. II. contains an account of the principles, on which this theory is founded ; *viz.* the universal gravitation of matter, and the diurnal rotation of the earth. Our author mentions likewise the hypothesis of the earth's being originally in a fluid state ; but rejects it as precarious and improbable. He allows however, that, with regard to the present question, it is all one whether it was first a fluid or not, seeing the ocean is circumfused just in the same manner, and to the same altitude, as if the whole was still a fluid.

10. Chap. III. and IV. are employed in the doctrine of centrifugal forces, and their effect in changing a fluid sphere into the form of an oblate spheroid. In the former of these chapters, the author resolves, as usual, the centrifugal force of a particle into two others ; one, that acts directly contrary to the gravitation of the particle ; and the other a force in a direction perpendicular to it. And this last he considers

B

again.

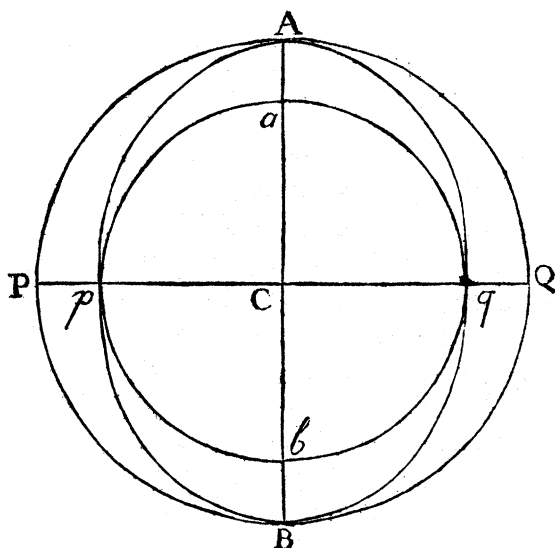
again as acting laterally upon the contiguous particles impelling them towards the equator. But the quantity of this force, when greatest at the octant, he computes to be only $\frac{1}{668\frac{1}{6}}$ of the force of gravity; and therefore, says he, it may be safely neglected. The truth is, after the spheroid is come to be in a permanent state, and all its parts in *æquilibrio*, there is no longer any such lateral force at all; it being now intirely satisfied by the gradual contraction of the earth's axis.

11. The general contents of the following chapters are sufficiently expressed in their titles already given. Nor can we be more particular, without entering into a detail of algebraical operations, which would be improper for this place; and which is the less necessary, as the same things have been treated of by several other authors.

This does not, however, in the least detract from the merit of F. Frisi; who discovers throughout this work much acuteness and skill, joined with all the candor and ingenuity, that become a philosopher. And as he has not yet exceeded his 23d year, it may be expected, that the sciences will one day be greatly indebted to him; especially as we find him actually engaged in composing a complete body of physico-mathematical learning.

12. There is only, in his VI. Chapter, a criticism upon one of Sir Isaac Newton's demonstrations, in which we cannot agree with this ingenious author. And as this demonstration has proved a stumbling-block, not only to F. Frisi, but to many other learned men, we shall be obliged to consider that part of it, which has been mistaken, at some length, by the help of the following scheme:

In



In which let the ellipsis $ApBqA$, whose axes AB , pq , are in any given ratio, as of m to n , have the circles $apbq$, and $APBQ$, inscribed and circumscribed to it: And if the figure revolves on the axis PQ , there will be generated an oblate spheroid $ApBqA$, with two spheres, the greater circumscribed to the spheroid, and touching it in its equator APA , and the lesser inscribed and touching it in the poles pq ; the solid content of the spheroid being the first of the two mean proportionals between the solidity of the exterior sphere, and that of the interior.

But if the figure revolves on the axis AB , there will be generated a prolate spheroid $ApBqA$, inscribed in the exterior sphere at the poles AB ; and circumscribing the interior sphere at the equator pqp , its solidity being the second of the above mean proportion-

als. So that if O and P stand for the solidities of the oblate and prolate spheroids, and S, s , for the two spheres; $S : O : P : s ::$ are in the continued proportion of $m : n$. And $S : P$, or $O : s :: m^2 : n^2$. As $S : s :: m^3 : n^3$.

Or we may with Sir Isaac Newton conceive of the genesis of these solids as follows. 1. Let the sphere $APBQ$ be uniformly compressed in the direction of its axis PQ , till that axis is diminished to pq , and the sphere changed into the oblate spheroid. 2. Let this spheroid be equally compressed in the direction of that diameter of its equator, which is perpendicular to pq and AB , or to the plane of the figure; and it will degenerate into the prolate spheroid, whose poles are A and B . 3. Let this last be compressed in the direction of its axis AB , till it is changed into the sphere $apbq$; and, in each of these compressions, the solid space, which the body contains will be diminished in the ratio of m to n .

Now, as the determination of the earth's figure depends not only upon that of the ratio of the centrifugal force, by which a body tends to recede from the axis of rotation to the power of gravity; but likewise, upon the decrement of gravitation arising from the body's being in that rotation actually removed to a greater distance from the centre: it is not enough, that we know, from the experiments with pendulums, the centrifugal force at the equator to be about $\frac{1}{180}$ of the force of gravity. We need, farther, two distinct propositions; one to determine the attractive force of a spheroid at its pole; and the other to determine its attraction at the equator. The first of these we have in *Princip. lib. 1. prop.*

91. and the second has been supplied by several authors. But Sir Isaac, who seldom does any thing in vain, found, that he could, by one of his artifices, make that 91st proposition serve likewise to determine the attraction at the equator, by the following argument :

Let G be the attraction of the exterior sphere at A ; and let the decrement of that attraction, when the sphere is diminished into the oblate spheroid $ApBq$, be d ; and δ the decrement of this last attraction, when the oblate spheroid is diminished into the prolate, whose poles are AB : then, I say, d is nearly equal to δ ; the difference of the axes of the generating ellipse being small.

For the attractive matter, that is taken away, has, in both cases, the same ratio to the matter, that is left; and its position, with respect to that which is left, is, in both cases, nearly the same: And therefore the successive attractions will be nearly in continued proportion, $G : G - d :: G - d + \delta ::$. Or multiplying and rejecting d^2 as inconsiderable, $Gd = G\delta$, and $d = \delta$.

Thus, if the attractions of the sphere $APBQ$, and of the prolate spheroid, at its pole A , be 126 and 125 respectively; the attraction of the intermediate oblate spheroid at its equator will be $125\frac{1}{2}$: and how nearly this approaches to the truth, may be seen from an exact computation of those attractions. For, if the axes of the generating ellipse be 101 and 100, and the attractive force at the surface of the sphere 126; the attraction at the pole of the prolate spheroid will be 124.9838; and that at the equator of the oblate 125.5077; which exceeds the arithmetical mean
between

between the two former, only by .0068 ; that is, by about $\frac{1}{18410}$ th part of the attraction of the oblate spheroid at the equator.

This reasoning is more shortly expressed (*Princip. lib. iii. Prop. 19.*) as follows:

.... " Gravitas in loco *A* in sphæroidem, con-
 " volutione ellipsæos (*ApBq*) circa axem *AB* de-
 " scriptam, est ad gravitatem in eodem loco *A* in
 " sphæram centro *C* radio *AC* descriptam, ut 125
 " ad 126. Est autem gravitas in loco *A* in terram
 " media proportionalis inter gravitates in dictam
 " sphæroidem et sphæram; propterea quod sphæra,
 " diminuendo diametrum *PQ* in ratione 101 ad 100,
 " vertitur in figuram terræ; et hæc figura, diminu-
 " endo in eadem ratione diametrum tertiam, quæ dia-
 " metris *AP, PQ* perpendicularis est, vertitur in dic-
 " tam sphæroidem; et gravitas in *A*, in utroque casu,
 " diminuitur in eadem ratione quam proxime."

In which the expression "*eadem ratione*" occurring a second time has misled F. Frisi and others, to think this last ratio to be likewise that of the axes, or of 101 to 100: Whereas the *identity* of ratio's here asserted is to be referred only to the words "*utroque casu*;" the ratio itself being not that of the axes, or of *m* to *n*; but the half of that ratio (whatever it is found to be by *Prop. 91. lib. i.*) which the attraction of the sphere has to the polar attraction of the inscribed spheroid.

This inadvertence, however, of his own F. Frisi charges upon Sir Isaac Newton; and files it up, as the sixth of the errors, which he says have been discovered in the *Principia*. . . . " Ita dum stabilitæ
 " in 19 *lib. iii. propositione* terrestrium axium pro-
 " portionis

“ portionis fulcimentum et patrocinium quærimus,
 “ aliud in propositione eadem *sophisma* sese offert,
 “ quod eorum, quæ in principiis mathematicis New-
 “ toni nacta (*i. e.* detecta) sunt hæcenus, sextum
 “ est, &c.” But we may take it off the file again ;
 and, for the present, leave the other five, till they are
 considered of at more leisure.

13. In his 10th and last chapter, our author sums
 up the evidence, and finds, that all the good observa-
 tions, that have been made, as well by pendulums, as
 by actual mensuration, concur with the theory, in
 making the ratio of the earth's axis and equatorial
 diameter to be as 229 to 230. This is indeed a suf-
 ficient confirmation of the theory of gravitation : But
 it must be observed, that the coincidence is not per-
 haps quite so perfect as F. Frisi imagines. That
 ratio corresponds well enough to the exactness, to
 which the first elements of the calculus can be ob-
 tained ; *the length of a second pendulum, and that of*
the earth's equatorial diameter, from which the cen-
 trifugal force ($\frac{1}{289}$) is deduced. But, if we suppose
 that force to be accurately $\frac{1}{289}$, and compute more
 rigorously, we shall find the ratio in question to be
 very nearly that of 225 to 226 ; agreeing still with
 the observations as well as can be desired ; and shew-
 ing, at the same time, the inimitable art of Sir Isaac
 Newton in the contrivance and use of approxima-
 tions ; seeing the strictest calculation raises the equa-
 tor not the third part of a mean geographical mile
 above what he had found by his method.

I sent F. Frisi's book to my ingenious and learned
 friend the reverend Mr. Murdock, Fellow of this
 Society, who has fully consider'd the question con-
 cerning

cerning the figure of the earth; and who, after having perused the book, and discover'd the above mistake of F. Frisi, sent me the above theorem, and its demonstration. He likewise sent me the following theorems, which, he says, he had communicated to M. de Bremond, in the year 1740, when he was translating his treatise on sailing: But M. de Bremond dying soon after, those, who had the care of publishing the translation, printed it incorrect in several places; particularly the theorems for the prolate spheroid: On which account, he says, if they are thought worth preserving, they may be inserted in the *Philosophical Transactions*.

Postscript.

Theorems for computing the ratio of the attractive force of a spheroid, at its pole or equator, to that of the inscribed or circumscribed sphere.

1. In an oblate spheroid, the ratio is,

$$\begin{array}{l} \text{Pole} \\ \text{Equator} \end{array} \left\{ \left[\begin{array}{l} 1 + \frac{1}{m^2 - 1} - \frac{m^2}{m^2 - 1} \frac{1}{2} \times A : \frac{1}{3} \\ \frac{m^2}{m^2 - 1} \frac{1}{2} \times A - \frac{1}{m^2 - 1} : \frac{2}{3} \end{array} \right] \right\}$$

2. In a prolate spheroid, the ratio is,

$$\begin{array}{l} \text{Pole} \\ \text{Equator} \end{array} \left\{ \left[\begin{array}{l} 1 - \frac{m^2}{m^2 - 1} + \frac{m}{m^2 - 1} \frac{1}{2} \times l : \frac{1}{2} \\ \frac{m^2}{m^2 - 1} - \frac{m}{m^2 - 1} \frac{1}{2} \times l : \frac{2}{3} \end{array} \right] \right\}$$

In

In which $m:1$, as the greater axis of the generating ellipse is to the lesser. A is a circular arc, to the radius 1, whose tangent is $\sqrt{m^2-1}$, or its reciprocal, if $m^2-1 \leq 1$. And l is the natural logarithm of S , S being the sine of the arc, whose co-sine is $\frac{1}{m}$, $\sqrt{m^2-1}$, and V the versed sine of the same arc.

Note, The two first theorems, by substituting t for $\sqrt{m^2-1}$, coincide with those of Mr. Maclaurin for the oblate spheroid, in his dissertation on the tides.

III. *A Letter from the Rev. Mr. George Costard, Fellow of Wadham-College, Oxford, to Dr. Bevis, concerning the Year of the Eclipse foretold by Thales.*

Dear Sir,

Read Jan. 25, 1753. **I** THANK you greatly for the use of the Petersburg Acts, while in London, where Bayer supposeth the eclipse foretold by Thales to the Ionians fell out the year before Christ 603. Since my return home, upon looking over some papers, that I had formerly drawn up on that subject, I find, that I had determined it to have been the very same year. I will not trouble you with the chronological arguments, on which I founded that determination; and therefore shall only transcribe so much of those papers as relates to calculation.

C

Riccioli